W3/2 Flat modules, Localization

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Cor 1.19 (1) Let $(E_i)_{i\in I}$ be a family of modules. Then

$$(\forall i \in I: \ E_i \text{ is flat}) \Leftrightarrow \bigoplus_{i\in I} E_i \text{ is flat.}$$

(2) Projective modules are flat. (In particular, free modules are flat).

Proof: (1) Exercise, use that $\otimes$ distributes over $\oplus$.

(2) Projective modules are direct summands of free modules. Using (1) twice, it suffices to show that A_A is flat.

$\forall M \in A\text{-Mod}: \quad A \otimes_A M \cong M$ via $a\otimes m \mapsto am$.

If $f: M \to N$ is a mono, then

$$\begin{tikzcd} & A\otimes M \arrow[r, "A\otimes f"] \arrow[d, "\wr"] & A\otimes N \arrow[d, "\wr"] \\ 0 \arrow[r] & M \arrow[r, "f"] & N \end{tikzcd}$$

commutes, hence $\ker(A\otimes f) = 0$. $\square$

Exm: $\mathbb{Q}_{\mathbb{Z}}$ is flat, not projective

[not projective: because it is injective, Exc 3, Set 3 $\Rightarrow$ not projective

flat: $\mu_n: \begin{cases} n\mathbb{Z}\otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q} \\ \underbrace{nx\otimes q}_{\rotatebox{90}{=}\ 1\otimes nxq} \longmapsto nxq \end{cases}$ has inverse $\mu_n^{-1}(q) = 1\otimes q$]

Remark In case $\mathrm{Hom}(M,-)$, $\mathrm{Hom}(-,M)$, $M\otimes -$ are not exact, one can describe the cokernel [kernel]. Leads to the long exact sequences of homological algebra, and the families of derived functors $\mathrm{Ext}^\bullet_A$, $\mathrm{Tor}^A_\bullet$

2. Localizations of Rings and Modules

Idea: Given a ring A adjoin some inverses (fractions), but e.g. not all.

Let A be a ring.

Def: $S \subseteq A$ is a multiplicatively closed set (= mc set), if

Def: $S \subseteq A$ is a multiplicatively closed set (= mc set), if $1 \in S$ and $\forall s, s' \in S$: $ss' \in S$.

Exm
- $S = \{1, a, a^2, \dots\}$ for $a \in A$,
- $S = A \setminus P$ for $P \in \mathrm{Spec}(A)$, more generally $S = A \setminus \bigcup_{P \in X} P$ for $X \subseteq \mathrm{Spec}(A)$.
- $S = \{\text{non-zero-divisors of } A\}$,
- $I \trianglelefteq A$, $S := 1 + I$

Given a mc set $S \subseteq A$, the localization $S^{-1}A$ of A at S is defined as follows:

- As set, $S^{-1}A := A \times S /_\sim$ where

$$(a,s) \sim (b,t) :\Leftrightarrow \exists u \in S: \ atu = bsu \quad (\Leftrightarrow at = bs \text{ if } u \text{ non-zero-divisor})$$

[Equivalence relation: $(a,s) \sim (a,s)$ ✓ $\quad (a,s) \sim (b,t) \Rightarrow (b,t) \sim (a,s)$ ✓

Suppose $(a,s) \sim (a', s')$, $(a', s') \sim (a'', s'')$

$\Rightarrow \exists u, v \in S$: $\quad as'u = a'su$ and $a's''v = a''s'v$

$$\Rightarrow \underset{\underset{(as'')\,\underline{s'uv}}{\parallel}}{as'us''v} = a'sus''v = a''s'vsu = (a''s)\,\underset{\underset{S}{\text{⋒}}}{\underline{s'uv}}$$

⚠ Can't cancel s'!

$\Rightarrow (a,s) \sim (a'', s'')$]

Notation $\frac{a}{s} := [(a,s)]_\sim$

Now $\frac{a}{s} \cdot \frac{a'}{s'} := \frac{aa'}{ss'}$, $\quad \frac{a}{s} + \frac{a'}{s'} := \frac{as' + a's}{ss'}$ make $S^{-1}A$ into a ring with zero $\frac{0}{1}$, one $\frac{1}{1}$.

[To check: $+, \cdot$ are well-defined, i.e., independent of the chosen representative and ring axioms are satisfied. Same as for fraction fields, no surprises anymore.]